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Correspondence between quantum mechanical and classical treatments of ion–electron collisions in a magnetic field

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Abstract

The energy transfer of a heavy ion due to electron–ion collisions in a magnetic field is treated within perturbation theory both quantum mechanically and classically. For the quantum treatment the unperturbed Landau levels are represented either in a Cartesian or in a cylindrical basis. In both the cases the classical limit of the quantum mechanical expression is investigated and compared to the energy transfer in a fully classical treatment. It is demonstrated that the classical limit is more transparent in the cylindrical coordinates than in the Cartesian ones as the first case allows a connection to a classical description in terms of the impact parameter.

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1. Introduction

The collisions of electrons with ions, the ion energy loss and related processes in a strongly magnetized plasma are important in studies of the transport phenomena (see, e.g., [1] and references therein), plasma heating and magnetic confinement of thermonuclear plasmas, as well as a test of the theoretical aspects of these phenomena. In addition, the electron–ion collisions in a strong magnetic field play an important role in the cooling of heavy-ion and antiproton beams by electrons (or positrons) [2–4].

Numerical and analytical calculations have been performed for classical binary collisions (BC) between magnetized electrons [5, 6] and for collisions between magnetized electrons and ions [7–13]. As an ion is much heavier than an electron, its uniform motion is only

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weakly perturbed by collisions with the electrons. Hence for the electron–ion collision the perturbation theory in the ion charge Z provides a useful information for small angle scattering. This has been done previously in first order in Z and for an ion at rest [10] as well as in second order for a uniformly moving ion [11–13]. The quantum mechanical BC theory has been developed within stationary perturbation theory and has been used for the ions at rest [14–17]. Its nonstationary counterpart has been developed recently in [18].

Within a quantum mechanical BC formalism in this paper we investigate the ion energy transfer due to collisions with electrons.

The interaction is considered as a time-dependent perturbation to the Landau states of the magnetized electrons while the heavy ion moves classically. We start with the unperturbed problem of a charged particle in a magnetic field which is formulated either in Cartesian (CAR) or in cylindrical (CYL) coordinates [19]. Within the CAR approach the electron state α is given by the parameters $\alpha = \{n, \sigma, q_y, q_z\}$ [19], where $\sigma = \pm 1/2$ is the spin variable, $n = 0, 1, 2, \ldots$ is the Landau level, q_z and q_y are the electron momentum components along and perpendicular to the external magnetic field $\mathbf{B} = B\mathbf{e}_z$, respectively. In this geometry the Schrödinger equation is reduced to the 1D equation for a shifted harmonic oscillator. In CYL the electron state $\alpha = \{\nu; n; \sigma; q_z\}$ ($\nu = 0; \pm 1; \pm 2 \ldots$ is the angular momentum) and corresponding wavefunctions differ from CAR. The momentum component q_y in CAR is replaced here by angular momentum ν. Because of the different electron states α in both representations the BC treatment will describe different elementary collision processes which finally result in different energy transfers. It is the main purpose to investigate the classical limit of these quantum mechanical expressions obtained from CAR and CYL, and to compare them with classical results. As the main outcome we will show that using CYL allows a connection to the classical energy transfer as a function of the impact parameter. The use of cylindrical coordinates is hence much more appropriate for the present scattering problem, as it makes the transition to the classical description much more transparent.

2. Quantum mechanical energy transfer

We consider the electron–ion BC in the presence of a quantizing homogeneous magnetic field $\mathbf{B} = B\mathbf{e}_z$. The projectile heavy ion of mass M and charge Ze ($-e$ is the electron charge) moves with velocity $\mathbf{v}_i$. We assume a mass of the ion $M \gg m$ such that a classical description of its motion with a rectilinear trajectory is applicable. We assume that the particles interact with the potential $-e\Phi_{\mathrm{ie}}(\mathbf{r} - \mathbf{v}_i t)$, where $\mathbf{r}$ and $\mathbf{v}_i t$ are the coordinates of the electron and ion, respectively. For charged particles the function $\Phi_{\mathrm{ie}}(\mathbf{r})$ can be expressed, for instance, by the Yukawa-type screened potential, $\Phi_{\mathrm{ie}}(\mathbf{r}) = Ze\exp(-r/\lambda)/4\pi\varepsilon_0 r$ (λ is the screening length), for application in plasmas (see [18] for more details).

Our starting point is the Schrödinger equation $\mathrm{i}\hbar\dot{\psi} = \hat{H}\psi$, $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$ with the time-dependent perturbation $\hat{H}_1(t) = -e\Phi_{\mathrm{ie}}(\mathbf{r} - \mathbf{v}_i t)$ and the Hamiltonian of a free electron

$$\hat{H}_0 = \frac{1}{2m}(\hat{\mathbf{p}} + e\mathbf{A})^2 + \hbar\omega_c\hat{\sigma}_z, \tag{2.1}$$

where $\omega_c = eB/m$ and $\hat{\sigma}_z$ are the cyclotron frequency and the spin operator, respectively.

We seek an approximate solution of the Schrödinger equation in which the interaction potential is considered as a perturbation. We start with the zero-order unperturbed eigenstates $\psi_\alpha^{(0)}$ in the Landau state α which are described by the zero-order Schrödinger equation $\hat{H}_0\psi_\alpha^{(0)} = \mathrm{i}\hbar\dot{\psi}_\alpha^{(0)}$. The unperturbed electron wavefunction can be represented as $\psi_\alpha^{(0)}(\mathbf{r}, t) = \psi_\alpha^{(0)}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\Omega_\alpha t}$ [19], where $\Omega_\alpha = E_\alpha/\hbar$, and E_α are the eigenvalues of the free particles.

We define the energy transfer of an electron which is initially in the αth Landau state as

$$\Delta E_\alpha = -e\int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}\mathbf{r}\rho_\alpha(\mathbf{r},t)[\mathbf{v}_i \cdot \mathbf{E}_{\mathrm{ext}}(\mathbf{r},t)]. \tag{2.2}$$

Here $\mathbf{E}_{\mathrm{ext}}(\mathbf{r},t) = -\nabla\Phi_{\mathrm{ie}}(\mathbf{r}-\mathbf{v}_i t)$ is the electrical field created by a moving ion, $\rho_\alpha(\mathbf{r},t) = |\psi_\alpha(\mathbf{r},t)|^2$ is the probability density for the electron in the αth state ($-e\rho_\alpha(\mathbf{r},t)$ is the charge density in αth state). Within first-order perturbation theory the electron probability density is given by the wavefunction of the free particle. Then the first-order electron energy transfer vanishes by the symmetry reason (see, e.g., [12, 18]). The second-order energy transfer is proportional to the first-order probability density $\rho_\alpha^{(1)}(\mathbf{r},t)$ which can be represented as

$$\rho_\alpha^{(1)}(\mathbf{r},t) = \psi_\alpha^{(0)}(\mathbf{r},t)\psi_\alpha^{(1)*}(\mathbf{r},t) + \psi_\alpha^{(0)*}(\mathbf{r},t)\psi_\alpha^{(1)}(\mathbf{r},t), \tag{2.3}$$

where $\psi_\alpha^{(1)}$ is the first-order perturbation of the electron wavefunction. This function is given by [18]

$$\psi_\alpha^{(1)}(\mathbf{r},t) = \frac{e}{\hbar}\int \mathrm{d}\mathbf{k}\,\Phi_{\mathrm{ie}}(\mathbf{k})\sum_\beta \psi_\beta^{(0)}(\mathbf{r},t)S_{\beta\alpha}(\mathbf{k})\frac{\mathrm{e}^{\mathrm{i}(\Omega_{\beta\alpha}-\omega)t}}{\Omega_{\beta\alpha}-\omega-\mathrm{i}0}. \tag{2.4}$$

Here the infinitesimal i0 in equation (2.4) guarantees the vanishing of the electron wavefunction at $t \to -\infty$. $\Omega_{\beta\alpha} = \Omega_\beta - \Omega_\alpha$, $\omega = \mathbf{k}\cdot\mathbf{v}_i$, $-e\Phi_{\mathrm{ie}}(\mathbf{k})$ is the Fourier transformed two-body interaction potential. The matrix $S_{\beta\alpha}$ is given by $S_{\beta\alpha}(\mathbf{k}) = \langle\beta|\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}|\alpha\rangle$ [20].

Using the relation $S_{\beta\alpha}(\mathbf{k}) = S^*_{\alpha\beta}(-\mathbf{k})$, and substituting equations (2.3) and (2.4) into equation (2.2) we arrive after some lengthy calculations at

$$\Delta E_\alpha = -\frac{(2\pi)^2 e^2}{\hbar}\int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'\,\Phi_{\mathrm{ie}}(\mathbf{k}')\Phi_{\mathrm{ie}}(\mathbf{k})(\mathbf{k}\cdot\mathbf{v}_i)\delta(\omega+\omega')\sum_\beta S_{\alpha\beta}(\mathbf{k})S_{\beta\alpha}(\mathbf{k}')\delta(\Omega_{\alpha\beta}-\omega), \tag{2.5}$$

where $\omega' = \mathbf{k}'\cdot\mathbf{v}_i$, $\delta(x)$ is the Dirac function. The last expression is the quantum mechanical electron energy transfer. Because of conservation of the total energy the ion energy transfer is $-\Delta E_\alpha$. Note that equation (2.5) was obtained for any unperturbed wavefunctions $\psi_\alpha^{(0)}(\mathbf{r})$. We shall apply this general expression for CAR and CYL.

2.1. Cartesian basis (CAR)

In a Cartesian basis the vector potential of the magnetic field $\mathbf{A}$ is given by its components $A_x = A_z = 0$ and $A_y = Bx$. The eigenstates are labelled by $\alpha = \{n, \sigma, q_y, q_z\}$, and the unperturbed electron wavefunction in the Landau state α is [19]

$$\psi_\alpha^{(0)}(\mathbf{r}) = \frac{1}{L\lambda_B^{1/2}(2^n n!\sqrt{\pi})^{1/2}}\,\mathrm{e}^{\mathrm{i}(q_y y+q_z z)}\,\mathrm{e}^{-\xi^2/2}H_n(\xi)u_\sigma \tag{2.6}$$

with $\xi = x/\lambda_B + q_y\lambda_B$. $\lambda_B = (\hbar/m\omega_c)^{1/2}$ is the magnetic length. A factor L^{-1} was introduced for normalization reason (L is the normalization length). In equation (2.6) u_σ is the spin wavefunction, H_n is the Hermite polynomial and the eigenvalues of the free particles E_α are given by

$$E_{n\sigma}(q_z) = \frac{\hbar^2 q_z^2}{2m} + \hbar\omega_c\left(n+\sigma+\frac{1}{2}\right). \tag{2.7}$$

Within the CAR treatment the matrix $S_{\beta\alpha}$ ($\mathbf{k}$) can be written in the form [20]

$$S_{\beta\alpha}(\mathbf{k}) = \left(\frac{2\pi}{L}\right)^2 \mathrm{i}^{n'-n}\delta_{\sigma\sigma'}\delta_{k_y+q_y;q_y'}\delta_{k_z+q_z;q_z'}\,\mathrm{e}^{-\mathrm{i}k_x\lambda_B^2(q_y+k_y/2)}\,\mathrm{e}^{\mathrm{i}(n-n')\theta}F_{nn'}(\zeta) \tag{2.8}$$

with $\zeta = k_\perp^2\lambda_B^2/2$ and $\tan\theta = k_y/k_x$. $k_\perp$ is the component of $\mathbf{k}$ perpendicular to $\mathbf{b} = \mathbf{B}/B$. The function $F_{nn'}(\zeta)$ $(n, n' \geqslant 0)$ is given by [20]

$$F_{nn'}(\zeta) = \left(\frac{n!}{n'!}\right)^{1/2} \zeta^{(n'-n)/2}\, \mathrm{e}^{-\zeta/2} L_n^{n'-n}(\zeta), \tag{2.9}$$

$F_{nn'}(\zeta) = (-1)^{n-n'} F_{n'n}(\zeta)$, and $L_n^{n'}(\zeta)$ are the generalized Laguerre polynomials.

Substituting equation (2.8) into equation (2.5) within CAR we finally obtain

$$\Delta E_\alpha^{\mathrm{CAR}} = \frac{(2\pi)^4 e^2}{\lambda_B^2 v_r m\omega_c L^2} \int \mathrm{d}\mathbf{k} |\Phi_{\mathrm{ie}}(\mathbf{k})|^2 (\mathbf{k}\cdot\mathbf{v}_i) \sum_{l=-n}^{\infty} F_{n;l+n}^2(\zeta)\, \delta\left(\zeta_l(\mathbf{k}) + \frac{k_z^2\lambda_B^2}{2}\omega_c\right). \tag{2.10}$$

Here $\zeta_l(\mathbf{k}) = l\omega_c + \mathbf{k}\cdot\mathbf{v}_r$, $\mathbf{v}_r = v_{e\|}\mathbf{b} - \mathbf{v}_i$ ($v_{e\|} = \hbar q_z/m$ is the electron unperturbed classical velocity component parallel to $\mathbf{b}$) is the classical relative velocity of the guiding centre of the electron helical motion with respect to the ion.

Now we consider the classical limit of equation (2.10). The classical regime can be realized by the two limits $\lambda_B^2 = \hbar/m\omega_c \to 0$ ($\zeta \to 0$) and $n = E_\perp/\hbar\omega_c \to \infty$ (large quantum numbers), where $E_\perp = m v_{e\perp}^2/2$ is the electron classical energy perpendicular to the magnetic field with $v_{e\perp} = \hbar q_y/m$. Besides we note that in this limit $n\zeta \to k_\perp^2 a^2/4$, where $a = v_{e\perp}/\omega_c$ is the electron cyclotron radius. The limit of the function $F_{n;l+n}^2(\zeta)$ at $\zeta \to 0$, $n \to \infty$ and $n\zeta \to k_\perp^2 a^2/4$ as found in appendix A is:

$$F_{n;l+n}^2(\zeta)\big|_{\lambda_B\to 0; n\to\infty} = J_l^2(k_\perp a) + \zeta\, Q_l(k_\perp a) + \mathrm{O}(\zeta^2), \tag{2.11}$$

where J_l is the Bessel function of the lth order. The first term in equation (2.11) gives the full classical limit. The second term gives a quantum correction. Also we should expand the δ-function in equation (2.10) for small $k_z^2\lambda_B^2/2$. We note that the first term of this expansion in the summand of equation (2.10) (zero-order term over length λ_B) which is proportional to $J_l^2(k_\perp a)\delta(\zeta_l(\mathbf{k}))$ vanishes due to the antisymmetrical behaviour of this expression with respect to the changes $l \to -l$ and $\mathbf{k} \to -\mathbf{k}$. The other nonvanishing terms contribute to the electron energy transfer which leads to

$$-\Delta E_\alpha^{\mathrm{CAR}} \to \frac{(2\pi)^4 e^2}{2m v_r L^2} \int \mathrm{d}\mathbf{k} |\Phi_{\mathrm{ie}}(\mathbf{k})|^2 (\mathbf{k}\cdot\mathbf{v}_i) \sum_{l=-\infty}^{+\infty} \delta(\zeta_l(\mathbf{k})) \times \left\{ \frac{(\mathbf{k}\cdot\mathbf{b})^2}{\zeta_l(\mathbf{k})} J_l^2(k_\perp a) + \frac{k_\perp^2}{2\omega_c}\left[J_{l+1}^2(k_\perp a) - J_{l-1}^2(k_\perp a)\right]\right\}. \tag{2.12}$$

The last expression is the classical limit of the quantum mechanical CAR energy transfer. In this case the classical energy transfer depends on $v_{e\perp}$, $v_{e\|}$ and $\mathbf{v}_i$. Note however that equation (2.12) diverges logarithmically at large $\mathbf{k}$ or at small distances where the perturbative treatment is not adequate. We should therefore introduce an upper cutoff $k_{\max}$ to avoid this divergency. In addition, for a pure unscreened Coulomb interaction a lower cutoff $k_{\min}$ should be introduced in equation (2.12). This procedure is equivalent to the screening of the Coulomb potential at large distances by plasma electrons.

2.2. Cylindrical basis (CYL)

We now switch to another quantum mechanical formulation for the electronic states in the external magnetic field. It is known that the Schrödinger equation for the free particle in the presence of external magnetic field may also be represented in cylindrical coordinates. We consider an electron with given angular momentum ν ($\nu = 0; \pm 1; \pm 2\ldots$) and momentum q_z along the magnetic field. If we choose the axial symmetric vector potential in the form

$\mathbf{A} = (1/2)[\mathbf{B} \times \mathbf{r}] = (B/2)(x\mathbf{e}_y - y\mathbf{e}_x)$ or $A_\varphi = B\rho/2$, $A_z = A_\rho = 0$ then the unperturbed Schrödinger equation has the following solutions [19]:

$$\psi_\alpha^{(0)}(\mathbf{r}) = \frac{1}{(2\pi L)^{1/2}\lambda_B}\left[\frac{n!}{2^{|\nu|}(|\nu|+n)!}\right]^{1/2} u_\sigma\, \mathrm{e}^{\mathrm{i}\nu\varphi}\, \mathrm{e}^{\mathrm{i}q_z z}(2\xi)^{|\nu|/2}\, \mathrm{e}^{-\xi/2} L_n^{|\nu|}(\xi). \tag{2.13}$$

Here $\alpha = \{\nu; n; \sigma; q_z\}, \xi = \rho^2/2\lambda_B^2, n = 0, 1, 2, \ldots, \rho, \varphi$ and z are the cylindrical coordinates. The energy of the free electrons in the αth CYL state is given by

$$E_\alpha = E_{n+\eta_\nu;\sigma}(q_z) = \frac{\hbar^2 q_z^2}{2m} + \hbar\omega_c\left(n + \sigma + \eta_\nu + \frac{1}{2}\right) \tag{2.14}$$

with $\eta_\nu = (\nu + |\nu|)/2$, $\mu_\nu = (|\nu| - \nu)/2$. At $\nu \leqslant 0$ the electron energy is the same as $E_{n\sigma}(q_z)$ in the CAR basis (2.7).

Using wavefunctions (2.13) we calculate the matrix elements [21]

$$S_{\alpha\beta}(\mathbf{k}) = \frac{2\pi}{L}\mathrm{i}^{\nu'-\nu}\delta_{\sigma\sigma'}\delta(q_z' - q_z + k_z)\, \mathrm{e}^{\mathrm{i}(\nu'-\nu)\theta}\Im_{nn';\nu\nu'}(k_\perp\lambda_B), \tag{2.15}$$

where $\beta = \{\nu'; n'; \sigma'; q_z'\}$. When the quantum numbers ν and ν' have different signs ($\nu\nu' \leqslant 0$) the function $\Im_{nn';\nu\nu'}(y)$ is given by

$$\Im_{nn';\nu\nu'}(y) = (-1)^{n+n'+\eta_\nu+\mu_{\nu'}} F_{n;n'+|\nu'|}\left(\frac{y^2}{2}\right) F_{n';n+|\nu|}\left(\frac{y^2}{2}\right), \tag{2.16}$$

where $F_{nn'}$ is defined by equation (2.9). In the opposite case when $\nu\nu' \geqslant 0$

$$\Im_{nn';\nu\nu'}(y) = (-1)^{\mu_{\nu'}+\mu_\nu} F_{nn'}\left(\frac{y^2}{2}\right) F_{n+|\nu|;n'+|\nu'|}\left(\frac{y^2}{2}\right). \tag{2.17}$$

The quantum-mechanical expression for the electron energy transfer has been derived above (see equation (2.5)). Substituting equation (2.15) into equation (2.5) and using the relations $\Im_{n'n;\nu'\nu}(y) = (-1)^{\nu'-\nu}\Im_{nn';\nu\nu'}(y)$ one finds

$$\begin{aligned}\Delta E_\alpha^{\mathrm{CYL}} = -\frac{(2\pi)^3 e^2}{\hbar L}\int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'\Phi_{\mathrm{ie}}(\mathbf{k}')\Phi_{\mathrm{ie}}(\mathbf{k})(\mathbf{k}\cdot\mathbf{v}_i)\delta(k_z + k_z')\delta((\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}_r) \\ \times \sum_{n'=0}^{\infty}\sum_{\nu'=-\infty}^{\infty}(-1)^{\nu'-\nu}\,\mathrm{e}^{\mathrm{i}(\nu'-\nu)(\theta-\theta')}\delta\left(\zeta_{\eta_\nu-\eta_{\nu'}+n-n'}(\mathbf{k}) - \frac{\hbar k_z^2}{2m}\right) \\ \times \Im_{nn';\nu\nu'}(k_\perp\lambda_B)\Im_{nn';\nu\nu'}(k_\perp'\lambda_B).\end{aligned} \tag{2.18}$$

The last expression together with equations (2.16) and (2.17) gives the final result for the quantum mechanical electron energy transfer within the CYL treatment. The classical regime can be realized by the following limits: $\lambda_B^2 \to 0$ and large quantum numbers $\nu \to \infty$ and $n \to \infty$. Here two different regimes with positive or negative angular momentum can be distinguished. In the first regime, $\nu \geqslant 0, \eta_\nu = \nu$, the energy of an electron in the αth Landau state is $E_{n+\nu;\sigma}(q_z)$. The transition to classical mechanics therefore requires that $\nu + n \to m v_{e\perp}^2/2\hbar\omega_c = a^2/2\lambda_B^2 \to \infty$, where a is the classical electron Larmor radius. As will be shown below the quantum number n (which is always positive) is associated with the classical impact parameter s according to the relation $n \to s^2/2\lambda_B^2 \to \infty$. It will be proven that the impact parameter within the quantum treatment must be quantized with the quantum numbers n. As the number ν is positive here it thus requires $s \leqslant a$ for $\nu \geqslant 0$. Using equations (2.16)–(2.18) we obtain for $\nu \geqslant 0$:

$$\Delta E_\alpha^{\mathrm{CYL}} = -\frac{(2\pi)^3 e^2}{\hbar L}\int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'\,\Phi_{\mathrm{ie}}(\mathbf{k}')\Phi_{\mathrm{ie}}(\mathbf{k})(\mathbf{k}\cdot\mathbf{v}_i)\delta(k_z+k_z')\delta((\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}_r)$$
$$\times\left\{\sum_{n'=-n}^{\infty}\Phi_{n'+n;n}(\zeta,\zeta')\sum_{\nu'=-\nu}^{\infty}(-1)^{\nu'}\,\mathrm{e}^{\mathrm{i}\nu'(\theta-\theta')}\right.$$
$$\times\delta\left(\zeta_{\nu'+n'}(\mathbf{k}')+\frac{\hbar k_z^2}{2m}\right)\Phi_{n+\nu;n'+\nu'+n+\nu}(\zeta,\zeta')$$
$$+\sum_{n'=-(n+\nu)}^{\infty}(-1)^{n'}\,\mathrm{e}^{\mathrm{i}n'(\theta-\theta')}\delta\left(\zeta_{n'}(\mathbf{k}')+\frac{\hbar k_z^2}{2m}\right)\Phi_{n'+n+\nu;n+\nu}(\zeta,\zeta')$$
$$\left.\times\sum_{\nu'=1+\nu+n'}^{\infty}(-1)^{\nu'}\,\mathrm{e}^{-\mathrm{i}\nu'(\theta-\theta')}\Phi_{n;\nu'+n}(\zeta,\zeta')\right\}, \tag{2.19}$$

where $\Phi_{pq}(\zeta,\zeta') = F_{pq}(\zeta)F_{pq}(\zeta')$, $\zeta' = k_\perp'^2\lambda_B^2/2$. In the classical limit this corresponds to the energy transfer with small impact parameters, $s \leqslant a$.

In the second regime with $\nu \leqslant 0$, $\eta_\nu = 0$. The energy of an electron in αth state is $E_{n\sigma}(q_z)$ and the transition to classical mechanics requires $n \to a^2/2\lambda_B^2 \to \infty$. Now with the classical impact parameter s is associated the quantum number $n-\nu = n+|\nu|$ according to the relation $n-\nu \to s^2/2\lambda_B^2 \to \infty$. The case of negative ν thus requires $s \geqslant a$.

It can be proved that in the classical limit both expressions for the energy transfer with $\nu \leqslant 0$ and $\nu \geqslant 0$ (equation (2.19)) are equivalent and therefore we consider here the classical limit only for equation (2.19) with positive ν. The resulting classical expression will be valid for $s \leqslant a$ as well as for $s \geqslant a$. The limit of the function $F_{\nu;l+\nu}(z)$ at $\nu \to \infty$, $z \to 0$ (but $\nu z \to$ const) is given in appendix A. Using also the Taylor expansion of the Dirac δ-function with respect to the small parameter λ_B^2 from equation (2.19) we obtain

$$-\Delta E_\alpha^{\mathrm{CYL}} \to \frac{(2\pi)^3 e^2}{2mL}\int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'\,\Phi_{\mathrm{ie}}(\mathbf{k}')\Phi_{\mathrm{ie}}(\mathbf{k})(\mathbf{k}\cdot\mathbf{v}_i)\delta(k_z+k_z')\delta((\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}_r)$$
$$\times J_0(q)\sum_{l=-\infty}^{\infty}(-1)^l\,\mathrm{e}^{\mathrm{i}l(\theta-\theta')}\delta(\zeta_l(\mathbf{k}'))H_l(\mathbf{k},\mathbf{k}'), \tag{2.20}$$

where $q = |\mathbf{k}_\perp + \mathbf{k}_\perp'|s$ and

$$H_l(\mathbf{k},\mathbf{k}') = J_l(x)J_l(y)\frac{(\mathbf{k}\cdot\mathbf{b})(\mathbf{k}'\cdot\mathbf{b})}{\zeta_l(\mathbf{k}')} + \frac{k_\perp k_\perp'}{\omega_c}$$
$$\times\left[\frac{l}{x}J_l(x)J_l'(y) + \frac{l}{y}J_l(y)J_l'(x) + \mathrm{i}\sin(\theta-\theta')J_l(x)J_l(y)\right]. \tag{2.21}$$

Here $x = k_\perp a$ and $y = k_\perp' a$. The zero-order term over λ_B^2 vanishes and did not contribute to equation (2.20). This term changes its sign under the exchange $\mathbf{k} \leftrightarrows \mathbf{k}'$ and should be omitted. For the same reason in equation (2.21) some further terms also vanish after integration over $\mathbf{k}$ and $\mathbf{k}'$.

Let us note that in contrast to CAR the classical limit of CYL energy transfer diverges only at large distances for an unscreened Coulomb interaction. The classical CYL expression (2.20) depends on $v_{e\perp}$, $v_{e\parallel}$ and $\mathbf{v}_i$, and in addition on the impact parameter s. In this sense the conformity between quantum and classical descriptions is more complete here than within CAR. Since the purely classical problem does not depend on the specific coordinate basis one can expect that two different limits, equations (2.12) and (2.20), correspond to the classical energy transfer averaged over different sets of parameters. In the next section, we consider

the corresponding classical problem and look for the connection with the classical limits of quantum mechanical energy transfers.

3. Classical second-order energy transfer

As a basis for further consideration, we recall that the ion classical energy transfer $\Delta E_i(\mathbf{R}_0, \varphi)$ during the collision with a magnetized electron should depend on the initial position $\mathbf{R}_0$ and the initial phase φ of the electron helical motion [12]. We also introduce the variable $\mathbf{s} = \mathbf{R}_{0\perp}^{(r)}$ which is the component of $\mathbf{R}_0$ perpendicular to the relative velocity $\mathbf{v}_r$ of the electron guiding centre. The vector $\mathbf{s}$ is the distance of closest approach of the guiding centre. Since the initial phase of electrons and the azimuthal angle ϕ of $\mathbf{s}$ are not observable in quantum mechanical treatment we therefore consider the φ, ϕ-averaged energy transfer. The first-order energy transfer, which is proportional to the ion charge vanishes due to symmetry after averaging with respect to φ and ϕ [12]. Hence the ion energy change receives a contribution only from higher orders. The second-order energy transfer $\Delta E_i^{(2)}$, which is proportional to Z^2, was found in [12]. Averaging this expression over the initial phase of the electrons results in

$$\left\langle \Delta E_i^{(2)} \right\rangle_\varphi = \frac{\pi \mathrm{i} e^2}{m} \int \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{k}'\, \Phi_{\mathrm{ie}}(\mathbf{k}) \Phi_{\mathrm{ie}}(\mathbf{k}')(\mathbf{k}\cdot\mathbf{v}_i)\delta((\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}_r)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_0} \times \sum_{l=-\infty}^{+\infty} (-1)^l\, \mathrm{e}^{-\mathrm{i}l(\theta'-\theta)} J_l(k_\perp a) J_l(k'_\perp a) G_l(\mathbf{k}, \mathbf{k}'), \tag{3.1}$$

where G_l is given by

$$G_l(\mathbf{k}, \mathbf{k}') = -\frac{2(\mathbf{k}\cdot\mathbf{b})(\mathbf{k}'\cdot\mathbf{b})}{(\zeta_l(\mathbf{k}') - \mathrm{i}0)^2} - \frac{\mathbf{k}\cdot(\mathbf{k}' - \mathbf{b}(\mathbf{k}'\cdot\mathbf{b}) - \mathrm{i}[\mathbf{b}\times\mathbf{k}'])}{(\zeta_l(\mathbf{k}') - \mathrm{i}0)(\zeta_{l-1}(\mathbf{k}') - \mathrm{i}0)} - \frac{\mathbf{k}\cdot(\mathbf{k}' - \mathbf{b}(\mathbf{k}'\cdot\mathbf{b}) + \mathrm{i}[\mathbf{b}\times\mathbf{k}'])}{(\zeta_l(\mathbf{k}') - \mathrm{i}0)(\zeta_{l+1}(\mathbf{k}') - \mathrm{i}0)}. \tag{3.2}$$

We consider now some averaged quantities which can be obtained from expression (3.1) for the energy transfer. First, the φ-averaged ion energy change, equation (3.1), is integrated over the 2D impact parameter $\mathbf{s}$ in the full space. Then one obtains that the classical limit (2.12) corresponds to

$$-\Delta E_\alpha^{\mathrm{CAR}} \to \frac{1}{L^2}\mathcal{E}_1 \equiv \frac{2\pi}{L^2}\int_0^\infty \left\langle \Delta E_i^{(2)} \right\rangle_{\varphi;\phi} s\, \mathrm{d}s. \tag{3.3}$$

We now average the φ-averaged ion energy change over the variable $z_0 = \mathbf{b}\cdot\mathbf{R}_0$, which is the component of the vector $\mathbf{R}_0$ along the magnetic field, and with respect to the angle ϕ. This gives the classical limit of the CYL energy transfer

$$-\Delta E_\alpha^{\mathrm{CYL}} \to \frac{1}{L}\mathcal{E}_2(s) \equiv \frac{1}{L}\int_{-\infty}^\infty \mathrm{d}z_0 \left\langle \Delta E_i^{(2)} \right\rangle_{\varphi;\phi}. \tag{3.4}$$

Equations (3.3) and (3.4) for the averaged energy changes $\mathcal{E}_1$ and $\mathcal{E}_2(s)$ have a different physical meaning. The quantity $\mathcal{E}_1$ is proportional to the transport cross section for the classical electron–ion scattering in the presence of an external magnetic field [12]. The second quantity $\mathcal{E}_2(s)$ is the classical energy transfer averaged over the all initial positions z_0 of the electron guiding centre. Therefore from the quantum mechanical treatment only the φ, ϕ, s (in CAR) or φ, ϕ, z_0 (in CYL) averaged classical energy transfers can be recovered because in quantum mechanics only these quantities are observable and the electron angular orientation and initial position of the guiding centre cannot be fixed.

It should be noted that although the full quantum mechanical expressions (2.10) and (2.18) may strongly differ from each other one can expect that for many-electron systems the statistical averaged physical quantities are the same for both treatments. In particular, we briefly show below the equivalence of the dielectric response functions obtained within the CAR and CYL formulations.

For any coordinate basis the dielectric function is given by [20]

$$\varepsilon(\mathbf{k},\omega) = 1 + \frac{e^2}{\varepsilon_0 k^2}\sum_{\alpha;\beta}|S_{\beta\alpha}(\mathbf{k})|^2\frac{f(E_\alpha)-f(E_\beta)}{E_\beta - E_\alpha - \hbar\omega - \mathrm{i}0}. \tag{3.5}$$

Here the arguments of the Fermi–Dirac function $f(E)$ are given by the eigenvalues of the free particles, equation (2.7) in CAR or equation (2.14) in CYL. The matrix elements $S_{\beta\alpha}(\mathbf{k})$ are defined by equations (2.8) and (2.15), respectively. Equation (3.5) has been evaluated in [20] for CAR. We now apply equation (3.5) for calculation of the dielectric function in CYL. Using equation (2.15) and the expansion of the 2D Dirac function with respect to the orthogonal functions $F_{\nu\nu'}$,

$$\sum_{\nu;\nu'=0}^{\infty}\mathrm{e}^{\mathrm{i}(\nu-\nu')(\theta-\theta')}F_{\nu\nu'}(\zeta)F_{\nu\nu'}(\zeta') = \frac{2\pi}{\lambda_B^2}\delta(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{3.6}$$

we arrive at the known expression for the dielectric function of the magnetized quantum plasma obtained within CAR treatment [20]. This shows the complete conformity between the CAR and CYL approaches.

4. Conclusions

In the present paper we have studied the electron–ion binary collisions in a quantizing magnetic field. Using time-dependent perturbation theory the second-order energy transfer is calculated within two different representations of the Landau wavefunctions, namely in Cartesian (CAR) and cylindrical (CYL) coordinates. In CAR the electron state is given by the set of parameters $\{n, \sigma, q_y, q_z\}$ which includes the spin variable σ, the Landau level n and the electron momentum components along and perpendicular to the external magnetic field, q_z and q_y respectively. In this geometry the Schrödinger equation is reduced to the 1D harmonic oscillator equation. The CYL representation is given by the set of parameters $\alpha = \{\nu; n; \sigma; q_z\}$ with the angular momentum ν. The Schrödinger equation is reduced to the 2D harmonic oscillator equation and its solution essentially differs from the CAR. Using the CAR and CYL approaches we have derived the classical limits for the corresponding energy transfers. Due to the differences in the spatial symmetries the CAR and CYL energy transfers lead to different classical results. In the classical limit $\Delta E_\alpha^{\mathrm{CAR}}$ gives the transport cross-section and does not depend on the classical impact parameter. The second energy transfer, $\Delta E_\alpha^{\mathrm{CYL}}$, represents the classical energy transfer doubly averaged with respect to the phase and the distance along the trajectory of the electron guiding centre. It depends on the impact parameter s. Moreover it has been shown that for $\nu \geqslant 0$ the role of the quantum mechanical 'impact parameter' plays the quantum number n according to $n \to s^2/2\lambda_B^2$. For $\nu < 0$ we obtain $n - \nu \to s^2/2\lambda_B^2$. Obviously, the CYL treatment displays the transition to the classical limit in more detail than CAR. Finally, we have shown the identity of the quantum dielectric functions obtained within CAR and CYL treatments.

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Appendix. Classical limit for the function $F_{nn'}(\zeta)$

In this appendix we give a detailed derivation of the classical limit of the function $F_{n;l+n}(\zeta)$ with fixed l and $n \to \infty, \zeta \to 0$ but $n\zeta = c^2/4 = \text{const}$. We express the Laguerre polynomials through the confluent hypergeometric sum [21] which results for the functions $F_{n;l+n}(\zeta)$, using equation (2.9), in

$$F_{n;l+n}(\zeta) = \left[\frac{\Gamma(n+l+1)}{\Gamma(n+1)}\right]^{1/2} \zeta^{l/2}\, \mathrm{e}^{-\zeta/2} \sum_{k=0}^{n} \frac{\Gamma(-n+k)}{\Gamma(-n)} \frac{1}{\Gamma(k+l+1)} \frac{\zeta^k}{k!}, \tag{A.1}$$

where $\Gamma(x)$ is the Euler function. For forthcoming consideration the following limit will be useful:

$$\left.\frac{\Gamma(q+k)}{q^k \Gamma(q)}\right|_{q\to\infty} = 1 + \frac{k^2-k}{2q} + \mathrm{O}\left(\frac{1}{q^2}\right). \tag{A.2}$$

Here k is an arbitrary but fixed number. Using the last relation and the standard representation of the Bessel functions [21] in the limits $n \to \infty, \zeta \to 0$ but $n\zeta = c^2/4 = \text{const}$ equation (A.1) becomes

$$F_{n;l+n}(\zeta)|_{n\to\infty;\,\zeta\to 0} = J_l(c) + \zeta P_l(c) + \mathrm{O}(\zeta^2), \tag{A.3}$$

where $P_l(c) = ((l+1)/c)J_l'(c)$. The function Q_l in equation (2.11) is given by $Q_l(c) = 2J_l(c)P_l(c)$.

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